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Non-static nuclear forces in a Kerr–Newman background space

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Abstract. In the Kerr–Newman background space, an explicit expression for the source term due to a particle moving along a geodesic near the event horizon in the equatorial plane of the black hole is found. This is used, together with the solutions of the Klein–Gordon equation near the event horizon (found elsewhere) to show that the meson field near the black hole vanishes as the source crosses the event horizon.

1. Introduction

In a recent series of papers (see Rowan and Stephenson 1976a, b, 1977 and Rowan 1977), the Klein–Gordon equation for a massive scalar meson field has been examined in various background spaces. Rowan (1977) has extended the work of Rowan and Stephenson to the in-fall of an uncharged baryon down the axis of a charged rotating black hole described by the Kerr–Newman metric, and has shown that the field of the baryon source falls to zero as the source crosses the event horizon. By allowing the particle to move down the axis of rotation, Rowan was able to treat the in-fall as a series of quasi-static problems since the event horizon and the static limit coincide on the axis of rotation.

In this paper we extend this work to the in-fall of a baryon along a geodesic in the equatorial plane of the black hole. This requires that the source term be modified to a time-dependent one, since the tidal forces inside the ergosphere destroy the static situation. By solving the geodesic equations near the event horizon and using the solution of the Klein–Gordon equation near the event horizon as found by Rowan and Stephenson (1977), we have again deduced that the field of the baryon falls off to zero as the particle crosses the event horizon. It has not been possible to solve the basic equation over the whole range owing to the breakdown of the uniform asymptotic method. The reason for this will emerge in the following analysis.

2. Basic equations

We start with the Klein–Gordon equation

$$(\Box^2+\mu^2)\Phi=4\pi f(t,r,\theta,\phi) \tag{2.1}$$

where Φ is the scalar field and $f(t,r,\theta,\phi)$ represents a point source. In generally covariant form (2.1) is

$$\frac{1}{\sqrt{-g_4}}\frac{\partial}{\partial x^i}\left((\sqrt{-g_4})g^{ik}\frac{\partial\Phi}{\partial x^k}\right)+\mu^2\Phi=4\pi f(t,r,\theta,\phi). \tag{2.2}$$

Together with the Kerr–Newman metric in Boyer–Lindquist coordinates

$$\mathrm{d}s^2=\frac{\Delta}{\rho^2}(\mathrm{d}t-a\sin^2\theta\,\mathrm{d}\phi)^2-\frac{\sin^2\theta}{\rho^2}[(r^2+a^2)\,\mathrm{d}\phi-a\,\mathrm{d}t]^2-\frac{\rho^2}{\Delta}\mathrm{d}r^2-\rho^2\,\mathrm{d}\theta^2 \tag{2.3}$$

where

$$\Delta=r^2-2Mr+a^2+Q^2,\qquad \rho^2=r^2+a^2\cos^2\theta, \tag{2.4}$$

equation (2.2) becomes

$$\left[\frac{[(r^2+a^2)^2-\Delta a^2\sin^2\theta]}{\Delta}\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial r}\left(\Delta\frac{\partial}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{(\Delta-a^2\sin^2\theta)}{\Delta\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right.$$
$$\left.-\frac{2a[\Delta-(r^2+a^2)]}{\Delta}\frac{\partial^2}{\partial\phi\,\partial t}+\rho^2\mu^2\right]\Phi=4\pi\rho^2 f(t,r,\theta,\phi). \tag{2.5}$$

Write

$$\Phi=\sum_{l,m}\int\mathrm{d}\omega\,(R_{lm\omega}(r)S_l^m(\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\,\mathrm{e}^{-\mathrm{i}\omega t}) \tag{2.6}$$

where $S_l^m(\theta)=S_l^m(a^2(\mu^2-\omega^2),\cos\theta)$ is the oblate spheroidal harmonic satisfying

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)+\lambda_{lm}-a^2(\mu^2-\omega^2)\cos^2\theta-\frac{m^2}{\sin^2\theta}\right]S_l^m(\theta)=0 \tag{2.7}$$

and λ_{lm} is the eigenvalue corresponding to $S_l^m(\theta)$. Taking the normalisation

$$\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta|S_l^m(\theta)|^2=1 \tag{2.8}$$

and substituting (2.6)–(2.8) into (2.5), we see that $R_{lm\omega}(r)$ satisfies

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{a^2m^2+2am\omega(Q^2-2Mr)+(r^2+a^2)^2\omega^2}{\Delta}-\lambda_{lm}-a^2\omega^2-\mu^2r^2\right]R_{lm\omega}(r)$$
$$=-2\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\rho^2\sin\theta S_l^m(\theta)\,\mathrm{e}^{-\mathrm{i}m\phi}f(t,r,\theta,\phi)\,\mathrm{d}\theta. \tag{2.9}$$

3. The geodesic equations

We take the equations of motion along a geodesic in a Kerr–Newman background space (Misner *et al* 1973) and consider the case of motion confined to the equatorial plane of a black hole. The equations become

$$r^2\frac{\mathrm{d}r}{\mathrm{d}\lambda}=\sqrt{R},$$
$$r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=-(aE-L_z)+\frac{aP}{\Delta}, \qquad (3.1)$$
$$r^2\frac{\mathrm{d}t}{\mathrm{d}\lambda}=-a(aE-L_z)+\frac{(r^2+a^2)P}{\Delta},$$

where

$$P=E(r^2+a^2)-L_za,$$
$$R=P^2-\Delta[\bar{\mu}^2r^2+(L_z-aE)^2] \qquad (3.2)$$

and where $\bar{\mu}$ is the rest mass of the baryon, and E and L_z are the energy at infinity and the angular momentum about the axis of rotation, respectively.

Putting $G=L_z-aE$, we get from (3.2)

$$P=Er^2-aG,$$
$$R=(Er^2-aG)^2-\Delta(\bar{\mu}^2r^2+G^2). \qquad (3.3)$$

From (3.1) using (3.3) we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=\frac{G\Delta+a(Er^2-aG)}{aG\Delta+(r^2+a^2)(Er^2-aG)} \qquad (3.4)$$

and

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\Delta[(Er^2-aG)^2-\Delta(\bar{\mu}^2r^2+G^2)]^{1/2}}{aG\Delta+(r^2+a^2)(Er^2-aG)}. \qquad (3.5)$$

We now confine our attention to motion near the event horizon $r=r_+=M+[M^2-(a^2+Q^2)]^{1/2}$ and assume that $a^2+Q^2\neq M^2$ so that $r_+\neq r_-=M-[M^2-(a^2+Q^2)]^{1/2}$.

Putting

$$Mx=r-r_+, \qquad 2Md=r_+-r_- \qquad (3.6)$$

we have

$$\Delta=M^2x(x+2d). \qquad (3.7)$$

Substituting (3.6) and (3.7) into (3.4) and (3.5), we may expand the right-hand sides in powers of x to get

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=\frac{a}{(r_+^2+a^2)}+\alpha x+\mathrm{O}(x^2) \qquad (3.8)$$

and

$$\frac{\mathrm{d}r}{\mathrm{d}t}=M\frac{\mathrm{d}x}{\mathrm{d}t}=-\left(\frac{2dM^2x}{(r_+^2+a^2)}+\beta x^2\right)+\mathrm{O}(x^3), \qquad (3.9)$$

taking the minus square root in (3.5), and where the constants α and β are given by

$$\alpha = \frac{2M}{(r_+^2 + a^2)^2 (Er_+^2 - aG)} (MG\, dr_+^2 - aEr_+^3 + a^2 G r_+) \tag{3.10}$$

and

$$\beta = \frac{M^2}{(r_+^2 + a^2)} \left(\frac{Md(4E^2 r_+^3 - 4aGEr_+ - 2Md\bar{\mu}^2 r_+^2 - 2MdG^2)}{(Er_+^2 - aG)^2} - \frac{2Md(4Er_+^3 + 2a^2 Er_+ - 2aGr_+ + 2aGMd)}{(r_+^2 + a^2)(Er_+^2 - aG)} + 1 \right) \tag{3.11}$$

for $Er_+^2 - aG \neq 0$.

4. The source term

To get an explicit expression for $f(t, r, \theta, \phi)$, we choose, following Persides (1974),

$$f(t, r, \theta, \phi) = g \frac{1}{u^0} \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}') \tag{4.1}$$

where $u^0 = dt/ds$ along the trajectory of the particle $\boldsymbol{r}'(t) = (r'(t), \theta'(t), \phi'(t))$ and g is the source strength. To calculate $1/u^0$ we first put (3.6) and (3.7), together with $\theta = \pi/2$, into the metric (2.3), obtaining

$$\left(\frac{ds}{dt}\right)^2 = \frac{M^2 x(x+2d)}{(Mx + r_+)^2} \left(1 - a\frac{d\phi}{dt}\right)^2 - \frac{1}{(Mx + r_+)^2} \left([(Mx + r_+)^2 + a^2] \frac{d\phi}{dt} - a \right)^2 - \frac{(Mx + r_+)^2}{M^2 x(x+2d)} M^2 \left(\frac{dx}{dt}\right)^2. \tag{4.2}$$

Then substituting (3.8) and (3.9) into (4.2), we see that to the second order in x,

$$\left(\frac{1}{u^0}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \gamma x^2 \tag{4.3}$$

where the constant γ is given by

$$\gamma = \frac{(2M^2 r_+^2 - 8dM^3 r_+)}{(r_+^2 + a^2)^2} - \frac{(4adM^2 \alpha + 2r_+^2 \beta)}{(r_+^2 + a^2)} - \frac{1}{r_+^2} \left(\frac{2Mar_+}{(r_+^2 + a^2)} + (r_+^2 + a^2)\alpha \right)^2. \tag{4.4}$$

On substituting into (4.4) the expressions for α and β from (3.10) and (3.11), we find, after considerable algebra, that γ is given simply by

$$\gamma = \frac{4d^2 M^4 \bar{\mu}^2 r_+^4}{(r_+^2 + a^2)^2 (Er_+^2 - aG)^2} \tag{4.5}$$

so that

$$\frac{1}{u^0} = K(r - r_+) \tag{4.6}$$

where

$$K = \frac{2dM\bar{\mu} r_+^2}{(r_+^2 + a^2)(Er_+^2 - aG)}. \tag{4.7}$$

From (4.1), using (4.6), the source term can be written

$$f(t, r, \theta, \phi) = gK(r-r_+)\delta(r-r_0(\phi_0))\delta(\phi-\phi_0(t))\delta(\theta-\tfrac{1}{2}\pi) \tag{4.8}$$

where, from (3.8) and (3.9),

$$\phi_0(t) = \frac{at}{(r_+^2+a^2)} \tag{4.9}$$

$$r_0(\phi_0) = r_+ + \exp\left(-\frac{(r_+-r_-)}{a}\phi_0\right). \tag{4.10}$$

Hence the right-hand side of (2.9) becomes on substituting (4.8)

$$-2gK(r-r_+)\delta(r-r_0)\int_{-\infty}^{\infty} e^{i\omega t}\,dt\int_0^{2\pi} d\phi\int_0^{\pi}\rho^2\sin\theta S_l^m(\theta)\,e^{-im\phi}\delta(\phi-\phi_0)\delta(\theta-\tfrac{1}{2}\pi)\,d\theta. \tag{4.11}$$

After performing the θ-integration (4.11) becomes

$$-2gKr^2(r-r_+)S_l^m(\tfrac{1}{2}\pi)\delta(r-r_0)\int_{-\infty}^{\infty} e^{i\omega t}\,dt\int_0^{2\pi} e^{-im\phi}\delta(\phi-\phi_0(t))\,d\phi. \tag{4.12}$$

Now from (4.9)

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\omega t}\,dt\int_0^{2\pi} e^{-im\phi}\delta(\phi-\phi_0(t))\,d\phi &= \int_0^{2\pi} d\phi\, e^{-im\phi}\int_{-\infty}^{\infty} e^{i\omega t}\delta\left(\phi-\frac{at}{(r_+^2+a^2)}\right)dt \\ &= \frac{2\pi}{a}(r_+^2+a^2) \qquad \text{for } \omega = \frac{ma}{(r_+^2+a^2)}\end{aligned} \tag{4.13}$$

so that (4.12) becomes

$$-4\pi g\frac{K}{a}(r_+^2+a^2)r^2(r-r_+)S_l^m(\tfrac{1}{2}\pi)\delta(r-r_0). \tag{4.14}$$

Finally equation (2.9) becomes

$$\left[\frac{d}{dr}\left(\Delta\frac{d}{dr}\right)+\frac{a^2m^2+2am\omega(Q^2-2Mr)+(r^2+a^2)^2\omega^2}{\Delta}-\lambda_{lm}-a^2\omega^2-\mu^2r^2\right]R_{lm\omega}(r)$$
$$= -4\pi g\frac{K}{a}(r_+^2+a^2)r^2(r-r_+)S_l^m(\tfrac{1}{2}\pi)\delta(r-r_0). \tag{4.15}$$

5. The radial equation

Rowan and Stephenson (1977) have shown that after defining x and d by

$$Mx = r - r_+, \qquad 2Md = r_+ - r_- \tag{5.1}$$

and writing

$$R_{lm\omega}(x) = Z(x)[x(x+2d)]^{-1/2}, \tag{5.2}$$

substitution of (5.1) and (5.2) into (4.15), leads to (for $r \neq r_0$)

$$\frac{d^2Z}{dx^2}+\left[M^2(\omega^2-\mu^2)+\frac{1}{M^2}\left(\frac{A}{x^2}+\frac{B}{x}+\frac{C}{(x+2d)^2}+\frac{D}{(x+2d)}\right)\right]Z=0 \qquad (5.3)$$

where A, B, C and D are constants. It has not been possible to solve (5.3) over the whole range $0 \leqslant x < \infty$ due to the breakdown of the uniform asymptotic method. This is due to the fact that the method depends on the existence of a large parameter in the differential equation which we do not necessarily have in (5.3) since ω may be close to or equal to μ. However, we may use the solutions obtained by Rowan and Stephenson for $x \to 0$ (that is $r \to r_+$). These are

$$R_{lm\omega}(r) \sim \begin{cases} R_{(1)}(r)=\dfrac{M}{\Delta^{1/2}}\,e^{-(F^{1/2}/M)(r-r_+)}\left(\dfrac{2F^{1/2}}{M}(r-r_+)\right)^{\frac{1}{2}+\bar{m}} \\ R_{(2)}(r)=\dfrac{M}{\Delta^{1/2}}\,e^{-(F^{1/2}/M)(r-r_+)}\left(\dfrac{2F^{1/2}}{M}(r-r_+)\right)^{\frac{1}{2}-\bar{m}} \end{cases} \qquad (5.4)$$

where

$$\bar{m}^2=\frac{1}{4}-\frac{A}{M^2},$$
$$F=\left(M^2(\mu^2-\omega^2)-\frac{C}{4M^2d^2}-\frac{D}{2M^2d}\right) \qquad (5.5)$$

provided $F \neq 0$. The case of $F=0$ was treated separately and we will not repeat the solutions here.

We now integrate (4.15) across the singularity and impose continuity of $R_{lm\omega}(r)$ at $r=r_0$ to get

$$\Delta_0\left\{\frac{dR_{lm\omega}}{dr}\bigg|_{r_0+0}-\frac{dR_{lm\omega}}{dr}\bigg|_{r_0-0}\right\}=-4\pi g\frac{K}{a}(r_+^2+a^2)r_0^2(r_0-r_+)S_l^m(\tfrac{1}{2}\pi) \qquad (5.6)$$

where $\Delta_0=(r_0-r_+)(r_0-r_-)$.

Then for r_0 near r_+ we have

$$R_{lm\omega}(r)=4\pi g\frac{K}{a}(r_+^2+a^2)r_0^2(r_0-r_+)S_l^m(\tfrac{1}{2}\pi)\begin{cases} R_{(2)}(r_0)R_{(1)}(r) & r_+ \leqslant r \leqslant r_0 \\ R_{(1)}(r_0)R_{(2)}(r) & r_0 \leqslant r \end{cases} \qquad (5.7)$$

where $R_{(1)}$ and $R_{(2)}$ are given by (5.4). If we now let $r_0 \to r_+$ we see from (5.7) that $R_{lm\omega}(r)$ (for $r_0 \leqslant r$) tends to zero since $R_{(1)}(r_0)$ either tends to zero if $\bar{m}$ is real, or is bounded if $\bar{m}$ is complex. Provided the series for Φ is uniformly convergent, $\Phi \to 0$ as $r_0 \to r_+$. We note that Φ here is an expression for the scalar field near the event horizon since (5.4) are solutions of (5.3) only for r near r_+.

6. Special cases

The case where $a^2+Q^2=M^2$ must be considered separately since $d=0$ and consequently from (3.9) the term of order x in dx/dt is zero. To simplify the algebra, we consider the case of an extreme Kerr black hole, so that $a=M$ and $Q=0$. Expanding $d\phi/dt$ to order x^2 and dx/dt to order x^4 and substituting these into (4.2) with $d=0$ and $a=M$ we find

$$\left(\frac{1}{u^0}\right)^2=\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2=\frac{\bar{\mu}^2M^2x^4}{4(EM-G)^2} \tag{6.1}$$

to order x^4. Hence

$$\frac{1}{u^0}=\frac{\bar{\mu}(r-M)^2}{2M(EM-G)} \tag{6.2}$$

which can be used in place of (4.6).

The homogeneous radial equation becomes

$$\frac{\mathrm{d}^2Z}{\mathrm{d}x^2}+\left(M^2(\omega^2-\mu^2)+\frac{\bar{A}}{x}+\frac{\bar{B}}{x^2}+\frac{\bar{C}}{x^3}+\frac{\bar{D}}{x^4}\right)Z=0 \tag{6.3}$$

where

$$R_{lm\omega}(x)=\frac{Z(x)}{x}, \qquad Mx=r-M \tag{6.4}$$

and the constants $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are given by

$$\begin{aligned}\bar{A}&=4M^2\omega^2-2M^2\mu^2\\ \bar{B}&=7M^2\omega^2-M^2\mu^2-\lambda_{lm}\\ \bar{C}&=8M^2\omega^2-4Mm\omega\\ \bar{D}&=4M^2\omega^2-4Mm\omega+m^2.\end{aligned} \tag{6.5}$$

The relation between ω and m is now

$$\omega=m/2M \tag{6.6}$$

and on substituting (6.6) into (6.5), we find

$$\begin{aligned}\bar{A}&=m^2-2M^2\mu^2\\ \bar{B}&=\tfrac{7}{4}m^2-M^2\mu^2-\lambda_{lm}\\ \bar{C}&=\bar{D}=0.\end{aligned} \tag{6.7}$$

Equation (6.3) now becomes

$$\frac{\mathrm{d}^2Z}{\mathrm{d}x^2}=\left(N-\frac{\bar{A}}{x}-\frac{\bar{B}}{x^2}\right)Z \tag{6.8}$$

where

$$N=M^2\mu^2-\tfrac{1}{4}m^2. \tag{6.9}$$

After defining

$$\eta=2N^{1/2}x \tag{6.10}$$

and substituting (6.10) into (6.8) we see that (6.8) has solutions in terms of Whittaker functions

$$Z=M_{\kappa,\pm\bar{m}}(\eta) \tag{6.11}$$

where

$$\kappa = \frac{\bar{A}}{2\sqrt{N}}, \qquad \bar{m}^2 = \tfrac{1}{4} - \bar{B}. \tag{6.12}$$

For the case $N = 0$ we get

$$Z = \begin{cases} x^{1/2} I_{\bar{\alpha}}(\bar{\beta} x^{1/2}) \\ x^{1/2} K_{\bar{\alpha}}(\bar{\beta} x^{1/2}) \end{cases} \tag{6.13}$$

where $I_{\bar{\alpha}}$, $K_{\bar{\alpha}}$ are the modified Bessel functions of order $\bar{\alpha}$ of the first and second kind respectively and

$$\bar{\alpha}^2 = 1 - 4\bar{B}, \qquad \bar{\beta}^2 = -4\bar{A}. \tag{6.14}$$

7. Conclusions

The success of the Liouville-Green asymptotic method when used to solve the radial equation for a massive scalar meson field (Rowan and Stephenson 1976a, b, Rowan 1977), and also when applied to the Schrödinger equation with a Gaussian potential (Stephenson 1977), depended on the appearance of a large parameter in the differential equation. This was due, in the first case, to the non-zero rest mass of the π-meson. When considering the most general black hole, solutions of the radial equation over the whole range are known only in special cases (Rowan 1977, Linet 1977); the equation may no longer contain a large parameter and in general will have four turning points. Although in principle it would be possible to match the solutions in the five regions, these problems together with the complexity of the differential equations for the geodesics give rise to great difficulties in any further work in this direction.

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